**Tax Voting Model**

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**Base Model**

Our base model was a simple, 2-good, 2-factor economy with 3 separate consumer groups. The two goods in the model are represented as goods X and Y, and the factors of production are K, used to denote capital, and L for labor. Then, on top of the model, we applied a consumption tax on good X.

Beyond the structure explained above, we made the following assumptions about our model: Consumer utility and production in both sectors were all modeled as Cobb-Douglas functions with constant returns to scale. Capital and labor endowments to consumer groups were determined exogenously. And total welfare for the economy was calculated as a Cobb-Douglas function, using the welfare of each individual consumer group as the inputs and their share of the total population as the share that they have of total welfare.

We then solved for all of these values using a general equilibrium model, satisfying the following conditions: Firms earn zero profits in the long-run, the markets for all goods and factors clear, and consumption of goods X and Y is equal to the consumer’s income. Then, a tax was applied to maximize the welfare of any one of the consumer groups or the total welfare for the economy. This was done using an MPEC solver, including the multiple complementarity conditions for the equilibrium model, then maximizing whatever welfare we chose subject to those conditions.

The code for these equations was as follows:

**\* ZERO PROFIT CONDITIONS**

**PRF\_X.. (1 + TAX) \* (PK\*(((BETA\*PL)/((1-BETA)\*PK))\*\*(1-BETA)) + PL\*((((1-BETA)\*PK)/(BETA\*PL))\*\*BETA))**

**=G= PX;**

**PRF\_Y.. PK\*(((GAMMA\*PL)/((1-GAMMA)\*PK))\*\*(1-GAMMA)) + PL\*((((1-GAMMA)\*PK)/(GAMMA\*PL))\*\*GAMMA)**

**=G= PY;**

**\* MARKET CLEARING CONDITIONS**

**MKT\_X.. X =G= SUM(I, SIZE(I)\*CONS(I)\*ALPHA(I)/PX);**

**MKT\_Y.. Y =G= SUM(I, SIZE(I)\*CONS(I)\*(1 - ALPHA(I))/PY);**

**MKT\_K.. SUM(I, KENDOW(I)\*SIZE(I)) =G= (X\*((BETA\*PL)/((1-BETA)\*PK))\*\*(1-BETA)) + (Y\*((GAMMA\*PL)/((1-GAMMA)\*PK))\*\*(1-GAMMA));**

**MKT\_L.. SUM(I, LENDOW(I)\*SIZE(I)) =G= (X\*(((1-BETA)\*PK)/(BETA\*PL))\*\*BETA) + (Y\*(((1-GAMMA)\*PK)/(GAMMA\*PL))\*\*GAMMA);**

**\* INCOME IDENTITY**

**I\_CONS(I).. CONS(I) =E= (LENDOW(I)\*PL + KENDOW(I)\*PK + SIZE(I)\*TAX\*X\*PX/(1+TAX));**

**\* WELFARE COMPUTATION**

**WELFARE(I).. W(I) =E= (((CONS(I)\*ALPHA(I)/PX)\*\*ALPHA(I)) \* ((CONS(I)\*(1 - ALPHA(I))/PY)\*\*(1-ALPHA(I))));**

**TOT\_WELFARE.. TOT\_WELF =E= PROD(I, W(I)\*\*SIZE(I));**

**WELFA.. WA =E= W("A");**

**WELFB.. WB =E= W("B");**

**WELFC.. WC =E= W("C");**

**MODEL EQUIL /PRF\_X.X, PRF\_Y.Y,**

**MKT\_X.PX, MKT\_Y.PY, MKT\_K.PK, MKT\_L.PL,**

**I\_CONS.CONS,WELFARE.W,TOT\_WELFARE,WELFA,WELFB,WELFC/;**

**Consumer Attributes**

Three groups of consumers exist in the model: group A, group B, and group C. Each group has a distinct proportion of the total population, endowment of labor, endowment of capital, and alpha parameter (a parameter in each group’s tailored Cobb-Douglas preference equation). The value of each of these parameters was assigned in order to develop three unique groups with unique sets of preferences.

Each group’s endowments of labor and endowment of capital determine a group’s respective income, measured in the **I\_CONS(I)** equation, which is a determinant in each group’s respective level of welfare.

The relative percentage of the total population included in each consumer group, **SIZE(I)**, directly affects the number of voters. The larger the consumer group, the more potential voters they have, and the more likely it is that the result of the voting process is in their favor. The size parameter also affects tax redistribution. A group with a larger proportion of the overall population receives a larger total share of the tax revenue, though an equal per capita share.

**Voter Preferences**

The policy being voted on within this model is the consumption tax rate on good X established within our base model. It works by having each consumer group propose a tax that would maximize their individual group’s welfare. Then, using the voting systems covered later in this paper, they are chosen as the result of a democratic vote.

Consumer voting preferences in all cases are determined within the model. The main assumptions that we made for this component were that all members of a consumer group would vote the same way, because they have identical attributes, and that they always vote for whichever available option maximizes their welfare. Further, the consumer groups will not employ any other strategy to game whatever voting system is being used; they will always vote for the option that best serves them, even if it has a miniscule chance of actually winning the vote.

**Calculating Voter Turnout**

Rather than assume 100% turnout by eligible voters, the model incorporates a multiplier value to determine each voting population’s propensity to vote. The value, labeled **VOTING(I)**, creates a more realistic and complex model by varying voter turnout with respect to voter preferences. This value is multiplied by each group’s population proportion, **SIZE(I)**, in order to calculate their number of voters.

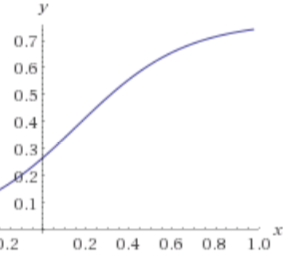
The bounds of the **VOTING(I)** value were determined by observing voter turnout rates in United States federal elections between 1948 and 2015. The lowest voter turnout observed was 39%, and the highest voter turnout observed was 65%. Rather than use this data to directly create the upper and lower bounds for **VOTING(I)**, the model allows for slightly more extreme levels of voter turnout; the lowest voter turnout possible for any given group in the model is 27%, and the highest voter turnout possible for any given group is 75%.

In order to calculate each group's voter turnout between these bounds, the model assumes that a voter’s marginal propensity to vote is not constant, but varies. The equation to determine voter turnout percentage is given by:

**VOTING(I) = 1/(1.3 + e (-4 \* (WELFGAP(I) – MINGAP) / (MAXGAP – MINGAP) + 0.9))**,

where **MINGAP** is the smallest difference between a group’s best perceived outcome from the voting process and the worst perceived outcome, and **MAXGAP** is the largest difference between a group’s best perceived outcome and the worst perceived outcome from the voting process. **(WELFGAP(I) – MINGAP) / (MAXGAP – MINGAP)** is a voter preference coefficient measured between 0 and 1 that factors into determining what percentage of each group turn out to vote.

The **VOTING(I)** equation looks like this:



It models voters with a non-constant marginal propensity to vote. The x value here comes from **(WELFGAP(I) – MINGAP) / (MAXGAP – MINGAP)**, and the y value is the turnout rate. A low value of **(WELFGAP(I) – MINGAP) / (MAXGAP – MINGAP)** results in a consumer group that feels as though the vote of a marginal voter will not make a noticeable difference in the outcome of the vote, therefore this group’s **VOTING(I)** value will be low. A high value of **(WELFGAP(I) – MINGAP) / (MAXGAP – MINGAP)** gives voters confidence that their consumer group will win the vote, regardless of whether the marginal voter votes or not. This gives the consumer group a high VOTING(I) value, but with a decreasing marginal propensity to vote. A **(WELFGAP(I) – MINGAP) / (MAXGAP – MINGAP)** value between 0.4 and 0.7 produces the highest marginal propensity to vote for a consumer group’s marginal voter, because they feel as if their vote could be the difference between winning the vote and coming in second or the difference between coming in second and finishing last in the vote.

Including the VOTING(I) equation adds voter reasoning to the model, rather than assuming all voters act the same way regardless of the effects of the tax plan their consumer group sponsors. Varying the marginal propensity to vote creates a more realistic model than setting it fixed.

**Voting Systems**

The first voting system we look at in the model is first-past-the-post. This system is very straightforward: all proposals are put forward and voted upon, and the one with the most votes wins the election. This system is used, for example, by most states in the U.S. during presidential elections, where the candidate with the most votes in the state receives that state’s electoral votes (the “winner take all” system).

To implement this in our model we use smax to check if the group with the most votes is group A: **If(smax(I,SIZE(I)\*VOTING(I)) = SIZE("A")\*VOTING("A"),**

If so, we store that they are the winner (**ACTUAL\_WINNER("FPPTURN") = 1**) and solve the model with their optimal tax: **SOLVE EQUIL USING MPEC maximizing WA.**

Otherwise, proceeding in the same way we check if the largest group is group B, and then if not it must be group C. Note that although this can produce misleading results if two groups receive exactly the same number of votes, this is a reflection of the indeterminacy of the voting system in that situation rather than an error in the model. The possibility can be removed in the model by adding a conditional to check if the winning group has the exact same number of votes as another (and then declaring no result), however, given distinct group sizes and the voter turnout function we chose, the probability of such an error is almost zero. Therefore, we omit a resolution here.

The full code for the first past the post system is here:

**If(smax(I,SIZE(I)\*VOTING(I)) = SIZE("A")\*VOTING("A"),**

**ACTUAL\_WINNER("FPPTURN") = 1;**

**SOLVE EQUIL USING MPEC maximizing WA;**

**Else If(smax(I,SIZE(I)\*VOTING(I)) = SIZE("B")\*VOTING("B"),**

**ACTUAL\_WINNER("FPPTURN") = 2;**

**SOLVE EQUIL USING MPEC maximizing WB;**

**Else**

**ACTUAL\_WINNER("FPPTURN") = 3;**

**SOLVE EQUIL USING MPEC maximizing WC;));**

Followed by code to store up the results for later display. We also run the same code without the voter turnout factor (that is, with 100% turnout for each group) for comparison.

The second system we have is instant runoff. In this system voting proceeds in stages until one proposal secures over 50% of the votes. At each stage the proposal with the fewest votes is eliminated.

To run this we first find and effectively eliminate the proposal with the fewest votes: **If (smin(I,VOTING(I)\*SIZE(I)) = VOTING("A")\*SIZE("A"),**

Then we look at which of the remaining proposals (B and C here) has more votes:

**If (VOTING("B")\*SIZE("B") + VOTING("A")\*SIZE("A")$(WW("B","A") > WW("C","A")) > VOTING("C")\*SIZE("C") + VOTING("A")\*SIZE("A")$(WW("C","A") > WW("B","A")),**

The votes for each side by their own group appear directly on the respective side of the inequality. The votes of the eliminated group (A here) are added to one side or the other based on which proposal gives them higher welfare, using the dollar operator.

Finally, as with first-past-the-post, we store the winner, solve the model at their optimal tax level, store the results, and do the same thing without voter turnout for comparison. The full code for the instant runoff is (results storing omitted):

**If (smin(I,VOTING(I)\*SIZE(I)) = VOTING("A")\*SIZE("A"),**

**If (VOTING("B")\*SIZE("B") + VOTING("A")\*SIZE("A")$(WW("B","A") > WW("C","A")) > VOTING("C")\*SIZE("C") + VOTING("A")\*SIZE("A")$(WW("C","A") > WW("B","A")),**

**ACTUAL\_WINNER("ROTURN") = 2;**

**SOLVE EQUIL USING MPEC maximizing WB;**

**ELSE**

**ACTUAL\_WINNER("ROTURN") = 3;**

**SOLVE EQUIL USING MPEC maximizing WC;)**

**ELSE IF (smin(I,VOTING(I)\*SIZE(I)) = VOTING("B")\*SIZE("B"),**

**If (VOTING("A")\*SIZE("A") + VOTING("B")\*SIZE("B")$(WW("A","B") > WW("C","B")) > VOTING("C")\*SIZE("C") + VOTING("B")\*SIZE("B")$(WW("C","B") > WW("A","B")),**

**ACTUAL\_WINNER("ROTURN") = 1;**

**SOLVE EQUIL USING MPEC maximizing WA;**

**ELSE**

**ACTUAL\_WINNER("ROTURN") = 3;**

**SOLVE EQUIL USING MPEC maximizing WC;)**

**ELSE**

**If (VOTING("A")\*SIZE("A") + VOTING("C")\*SIZE("C")$(WW("A","C") > WW("B","C")) > VOTING("B")\*SIZE("B") + VOTING("C")\*SIZE("C")$(WW("B","C") > WW("A","C")),**

**ACTUAL\_WINNER("ROTURN") = 1;**

**SOLVE EQUIL USING MPEC maximizing WA;**

**ELSE**

**ACTUAL\_WINNER("ROTURN") = 2;**

**SOLVE EQUIL USING MPEC maximizing WB;)**

**););**

**Voting System Criteria**

A number of criteria are used to evaluate different voting systems. By mathematically demonstrated results such as Arrow’s Impossibility Theorem, no voting system can hold every potentially desirable quality simultaneously, so there is a great deal of variation between different systems. In our model we look at two similar criteria, the Condorcet Winner and Condorcet Loser criteria.

The Condorcet winner is a proposal that would win a head-to-head (that is, a vote with only it and one other alternative) against any of the other proposals. In a voting system that satisfies the Condorcet Winner Criterion, a Condorcet winner must always win the general election (with all the alternatives on the table rather than just one at a time).

To check this in our model we set a new parameter corresponding to which group is the Condorcet winner, with none initially (a Condorcet winner does not necessarily exist in a given pool of proposals):

**PARAMETER Condorcet\_Winner(P);**

**Condorcet\_Winner("VOTE") = 0;**

Then we check if group A’s proposal would win a head-to-head against both B and C:

**Condorcet\_Winner("VOTE")$(VOTING("B")\*SIZE("B") + VOTING("C")\*SIZE("C")$(WW("B","C") > WW("A","C")) < VOTING("A")\*SIZE("A") + VOTING("C")\*SIZE("C")$(WW("A","C") > WW("B","C")) AND**

**VOTING("C")\*SIZE("C") + VOTING("B")\*SIZE("B")$(WW("C","B") > WW("A","B")) < VOTING("A")\*SIZE("A") + VOTING("B")\*SIZE("B")$(WW("A","B") > WW("C","B"))) = 1;**

This uses the dollar operator in two ways, first to make the assignment itself conditional (if the condition is not met, the **Condorcet\_Winner** parameter remains 0), and second as above for the instant runoff, to decide who the third group, whose proposal is not on the table, will vote for.

This is then repeated to check each group. There can be only one Condorcet winner (if one group wins every head-to-head, every other group must lose at least to them), so there is no issue with assigning one group and then switching in another group later on.

Neither first-past-the-post nor instant runoff meets this criteria. This is demonstrated in the results of our model (see below).

The Condorcet loser, similarly, is a proposal that would *lose* a head-to-head against any other. To satisfy this criterion, the voting system cannot have a Condorcet loser win the general election. We model this the same way as the Condorcet winner except with the inequality reversed:

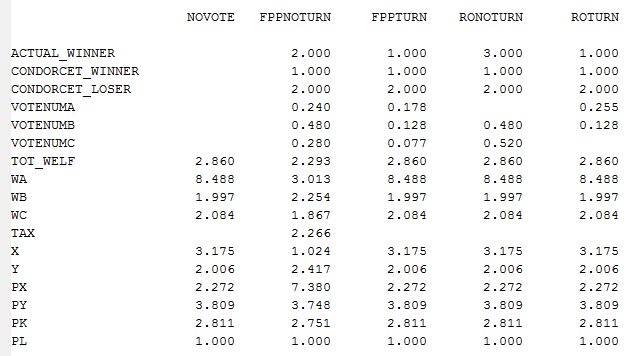
**Condorcet\_Loser("VOTE")$(VOTING("B")\*SIZE("B") + VOTING("C")\*SIZE("C")$(WW("B","C") > WW("A","C")) > VOTING("A")\*SIZE("A") + VOTING("C")\*SIZE("C")$(WW("A","C") > WW("B","C")) AND**

**VOTING("C")\*SIZE("C") + VOTING("B")\*SIZE("B")$(WW("C","B") > WW("A","B")) > VOTING("A")\*SIZE("A") + VOTING("B")\*SIZE("B")$(WW("A","B") > WW("C","B"))) = 1;**

Followed by checking for the other groups.

Instant runoff meets this criteria, since if a candidate can secure more than 50% of the vote at any time they cannot be a Condorcet loser (even if all the alternatives then pooled their votes behind one option, it would still have less than 50% of the vote and the first proposal would win that head-to-head). First-past-the-post however does not (see results below).

**Interpreting Output**

All the results in our model are displayed at the end of the listing file. They look like this: 

The columns correspond to the voting system: none (shows total welfare maximizing scenario), first-past-the-post first without then with voter turnout, and runoff without and with turnout. The first row shows which group wins the election under each voting system (1 = group A, 2 = group B, 3 = group C). The next two rows then show the Condorcet winner and losers. Vote numbers show how many people proportionally vote for each group’s proposal (1.0 = the entire population). In the runoff columns, these are the final round voting numbers. The group with 0 votes here is the one eliminated in the first round. TOT\_WELF is unsurprisingly total welfare, and WA/WB/WC give the per capita welfares of each group. Tax shows the tax rate (1.0 = 100% tax). X and Y are output levels of the respective good, followed by PX/PY/PK/PL for prices of X/Y/Capital/Labor respectively. PL is fixed at 1 as the numeraire.

Our initial values of parameters are

**ALPHA("A") = .9;**

**ALPHA("B") = .1;**

**ALPHA("C") = .2;**

**SIZE("A") = .24;**

**SIZE("B") = .48;**

**SIZE("C") = .28;**

**LENDOW("A") = 0;**

**LENDOW("B") = 10;**

**LENDOW("C") = 9;**

**KENDOW("A") = 10;**

**KENDOW("B") = 0;**

**KENDOW("C") = 1;**

**BETA = .25;**

**GAMMA = .75;**

These were chosen quasi-arbitrarily in order to demonstrate the results about the Condorcet criteria below. Group A has a strong preference for good X (**ALPHA = .9**), a large endowment of capital and no labor. They are the smallest group (**SIZE = .24)** Group B has the opposite preferences and endowments, and is the largest group. Group C is in between, but much closer to group B than group A in preferences and endowments. They are slightly larger than group A. Good X is labor intensive (**BETA = .25)**, while good Y is capital intensive (**GAMMA = .75)**.

There are three sets of results, one for each scenario we ran. The first, using the parameters and giving the results shown above, demonstrates how first-past-the-post without voter turnout does not meet either Condorcet criterion, and how runoff without turnout does not meet the Condorcet Winner Criterion. For the second we changed the group sizes:

**SIZE("A") = .21;**

**SIZE("B") = .6;**

**SIZE("C") = .19;**

This shows first-past-the-post *with* voter turnout violating both Condorcet criteria. Finally, the third uses:

**SIZE("A") = .20;**

**SIZE("B") = .56;**

**SIZE("C") = .24;**

It shows how runoff with voter turnout does not meet the Condorcet winner criterion.

Note that while in this case the outcomes were changed using the size parameters only, any or all of the other parameters can be varied too for different results.

**The Impact of Voter Turnout on the Model**

In the first set of results, we see that having voter turnout determined by how much they have to lose by not voting has a pretty significant impact on the outcome of a democratic vote. In the first case, consumer group B had a far larger population than both A and C that allowed them to win the first-past-the-post vote by a huge margin, and only lose the instant-runoff election by a small margin to the combined votes of groups A and C. However, after factoring in the voter turnout equation, we see that a huge number of members of the B group don’t show up, making it easier for group A to claim victory in both voting systems.

Even in the second set of results, there’s a similar outcome. In the first-past-the-post voting, Group B actually has a greater advantage, commanding 60% of the total population. But once voter turnout becomes a factor, most of the B voters fail to show up and they only win the election by a narrow margin. And because B’s majority disappears as a result of this, they actually end up losing the instant-runoff vote. This allows A and C to combine their votes and beat B by voting for A’s welfare-optimizing tax rate. The results for the third scenario also closely match this situation.

These different scenarios where voter turnout drastically alters the result of democratic voting demonstrates the power that a motivated voting base can have for certain politicians or policies, even if they make up a smaller chunk of the overall population. Funnily enough, in all of our scenarios, overall welfare was either improved or remained the same when voter turnout became a factor. This shows that by having certain people choose not to vote because they don’t care as much about the outcome, there can actually be some positive effects as a result, because the people that will be impacted by the policy to a larger degree will be given more of a voice in the decision.